ON THE ACCUMULATION OF DISTURBANCES IN LINEAR SYSTEMS

(O NAKOPLENII VOZMUSHCHENII V LINEINYKH SISTEMAKH)

PMM Vol. 25, No. 2, 1961, pp. 319-331

L. S. GNOENSKII
(Moscow)

(Received November 17, 1960)

1. The problem of determining at time T the maximum value of the quantity $y_{max}(T)$, a solution of the differential or difference equation $L_n(y) = f(t)$ under the conditions $|f(t)| \leqslant M_0$ on $[0, T]$, was considered in [1,2,3]. This problem is considered here under stronger restrictions for the right-hand side of the equation, namely, in the equation

$$L_n(y) \equiv y^{(n)} + a_1(t) y^{(n-1)} + \ldots + a_n(t) y = f(t) \tag{1.1}$$

the function $f(t)$ must satisfy the conditions

$$|f(t)| \leqslant M_0, \qquad |f'(t)| \leqslant M_1 \tag{1.2}$$

It is required to determine the function $f_m(t)$, satisfying (1.2) and providing the largest modulus to the solution $y(t)$ of the equation $L_n(y) = f_m(t)$ at time T. For definiteness it is assumed that

$$y(0) = y'(0) = \ldots = y_{(0)}^{(n-1)} = 0$$

A similar problem was formulated in [4] under certain restrictions imposed on the distance between the extrema of the function (1.5). Under analogous restrictions for difference equations, [5] treated a more general problem: it was assumed that

$$|f(t)| \leqslant M_0, \qquad |f'(t)| \leqslant M_1, \qquad |f''(t)| \leqslant M_2$$

The algorithm quoted in [6] allows one to construct a function $f_m(t)$ giving the modulus of $y(T)$, generally speaking an extremal but not the largest value. These restrictions are removed in the present work.

The problem formulated above is encountered in the design of control systems when: (1) in regard to the disturbances acting on the system it is known only that they are limited in the modulus and the derivative; (2) the application of rougher but simpler evaluations imposes prohibitively severe demands upon the parameters of the system; (3) the statistical characteristics of the disturbances are unknown or their application undesirable because of a responsibility of the system.

The problem of accumulation of disturbances is a particular case in the question of optimum control in L.S. Pontriagin's formulation. The difficulties arising in finding the maximal function $f_*(t)$ are connected with the existence of two restrictions upon the right-hand side of Equation (1.1). Therefore, this problem requires special consideration.

As is known, the solution $y(T)$ of Equation (1.1) can be expressed in the form

$$y(T) = \int_0^T G(T, t) f(t)\, dt \tag{1.3}$$

Let $f'(t) = \phi(t)$, then (1.3) can be expressed in the form

$$y(T) = \int_0^T F(t) \varphi(t)\, dt, \qquad F(t) = \int_t^T G(T, \tau)\, d\tau \tag{1.4}$$

The expression for $F(t)$ is obtained upon substitution of

$$f(t) = \int_0^t \varphi(\tau)\, d\tau \tag{1.5}$$

into (1.3) and change in the order of integration in the resulting double integral. Function $F(t)$ is continuous and differentiable, having on $[0, T]$ a finite number of extrema; modulus of $F'(t)$ is bounded on $[0, T]$.

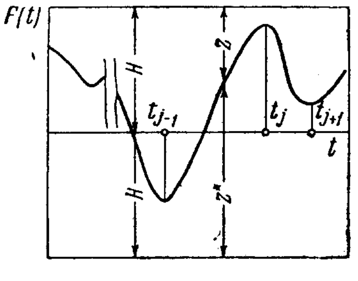

Fig. 1.

The above formulation can also be expressed as the following degenerate variational problem: find a function $\phi_*(t)$ belonging to a class of functions A satisfying on $[0, T]$ the conditions

$$|\varphi(t)| \leqslant M_1, \qquad \left|\int_0^t \varphi(t)\, dt\right| \leqslant M_0 \tag{1.6}$$

and giving the largest value to the functional

$$Y(\varphi) = \int_0^T F(t) \varphi(t)\, dt \tag{1.7}$$

2. Let us find the algorithm for the construction of a function $\phi_m(t)$ which we will call maximal. First let us introduce some notations. By $t_j (j = 2, \ldots, p)$ we will denote the extremum points of function $x = F(t)$, assuming $t_1 = 0$, $t_{p+1} = T$. For definiteness it is assumed that t_2 and consequently all t_j where j is even are maximum points of $F(t)$.

Let (Fig. 1)

$$H = \max F(t), \quad t \in [0, T], \quad z = H - F(t)$$
$$z^* = |-H - F(t)| = H + F(t)$$

For even j we introduce functions $t_{jr}(z)$ and $t_{jl}(z)$, while for odd j the functions $t_{jr}(z^*)$ and $t_{jl}(z^*)$.

Let j be even, then the quantities

$$t_{jl}(z) = t_j \quad \text{for } z \leqslant H - F(t_j)$$
$$t_{jl}(z) = t_{j-1} \quad \text{for } z \geqslant H - F(t_{j-1})$$
$$t_{jl}(z) \quad \text{for } H - F(t_j) < z < H - F(t_{j-1})$$

are the nearest, from the left of t_j, roots (Fig. 2) of the equation $H - z = F(t)$ relative to t.

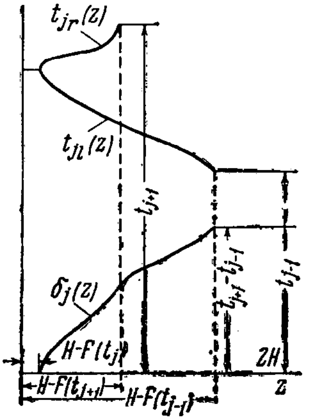

Fig. 2.

Further, the quantities

$$t_{jr}(z) = t_j \quad \text{for } z \leqslant H - F(t_j), \quad t_{jr}(z) = t_{j+1} \quad \text{for } z \geqslant H - F(t_{j+1})$$
$$t_{jr}(z) = \tau \text{ for } H - F(t_j) < z < H - F(t_{j+1})$$

are nearest, from the right of t_j, roots of the equation $H - z = F(t)$.

Let j be odd, then the quantities

$$t_{jl}(z^*) = tj \quad \text{for } z^* \leqslant H + F(t_j), \qquad t_{jl}(z^*) = t_{j-1} \quad \text{for } z^* \geqslant H + F(t_{j-1}),$$
$$t_{jl}(z^*) \quad \text{for } H + F(t_j) < z^* < H + F(t_{j-1})$$

are the nearest to the left of t_j, roots of the equation $H + F(t) = z^*$.

Furthermore

$$t_{jr}(z^*) = t_j \quad \text{for } z^* \leqslant H + F(t_j), \qquad t_{jr}(z^*) = t_{j+1} \quad \text{for } z^* \geqslant H + F(t_{j+1})$$

$$t_{jr}(z^*) \text{ for } H + F(t_j) < z^* < H + F(t_{j+1})$$

are the nearest, to the right of t_j, roots of the equation $H + F(t) = z^*$.

Assume

$$\delta_j(z) = t_{jr}(z) - t_{jl}(z), \qquad \delta_j(z^*) = t_{jr}(z^*) - t_{jl}(z^*)$$

Introduce functionals $\Phi_{ij}(z, z^*)$ dependent on the parameters u, where indices i, j may assume all possible integer values from zero to p, but for each of the functionals $i < j$.

Functionals $\Phi_{ij}(z, z^*)$ are defined by the sets of all positive, continuous, monotonically increasing, in a strict sense, functions $z = \gamma_1(u)$, $z^* = \gamma_2(u)$, where $\gamma_1(0) = \gamma_2(0) = 0$. If $z + z^* \leqslant 2H$, then

$$\Phi_{ij}(z, z^*) = \sum_{k=i+1}^{j}{}' \delta_k(z) - \sum_{k=i+1}^{j}{}'' \delta_k(z^*) \tag{2.1}$$

The first sum contains terms with even, while the second sum has terms with odd values of index k.

If the inequality

$$z_a + z_a^* = \gamma_1(u_a) + \gamma_2(u_a) = 2H$$

is valid for some u_a, then for $u > u_a$, and consequently for $z > z_a$, $z^* > z_a^*$

$$\Phi_{ij}(z, z^*) = \sum_{k=i+1}^{j}{}' \delta_k(z_a) - \sum_{k=i+1}^{j}{}'' \delta_k(z_a^*) \tag{2.2}$$

The first sum in (2.1) is equal to the sum of interval lengths between $t_{j+1,\,l}$ and t_{jr}, where $F(t) \geqslant z$; the second sum is equal to the sum of interval lengths between $t_{j+1,\,l}$ and t_{jr}, where $F(t) \leqslant z^*$.

The maximal function $\phi_m(t)$ is constructed stepwise with the aid of the functionals $\Phi_{ij}(z, z^*)$.

First step. Assume $z = z^* = u$ and let α_1 be the first value of u upon increase from zero, for which the inequality

$$|\Phi_{0j}(\alpha_1, \alpha_1)| = C_0$$

$$C_0 = \frac{M_0}{M_1}, \quad 0 \leqslant \alpha_1 \leqslant H$$

would be satisfied for at least one function $\Phi_{0j}(u, u)$, where $\Phi_{0j}(u, u)$ increases at a certain neighborhood to the right of the point $u = a_1$.

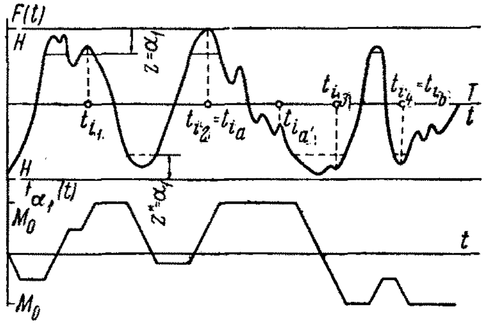

Fig. 3.

By $E_{a_1}^+$ we denote the system of intervals belonging to $[0, T]$ on which $F(t) > H - a_1$, while by $E_{a_1}^-$ we denote the system of intervals from $[0,T]$ on which $F(t) < -H + a_1$. Assume (Fig. 3)

$$\varphi_{\alpha_1}(t) = M_1 \quad \text{for } t \in E_{\alpha_1}^+, \qquad \varphi_{\alpha_1}(t) = -M_1 \quad \text{for } t \in E_{\alpha_1}^-$$
$$\varphi_{\alpha_1}(t) = 0 \quad \text{for } t \in [0, T] - E_{\alpha_1}^+ - E_{\alpha_1}^-$$

If $a_1 = H$, then $\phi_{a_1}^{\cdot}(t)$ is the maximal function, where $a_1 = H$. If $a_1 < H$, then $\phi_{*}(t)$ coincides with $\phi_{a_1}(t)$ only on the set

$$E_{\alpha_1} = E_{\alpha_1}^+ + E_{\alpha_1}^-$$

In the following steps $\phi_{*}(t)$ will be defined on the set $[0, T] - E_{a_1}$. Note that if $\Phi_{0j}(a_1, a_1) = C_0$ then (Fig. 3)

$$f_{\alpha_1}(t_{jr}) = \int_0^{t_{jr}} \varphi_{\alpha_1}(t)\, dt = M_1 \Phi_{0j}(\alpha_1, \alpha_1) = M_1 C_0 = M_0$$

Thus, in order to carry out the first step it is necessary to construct functions $\Phi_{0j}(u, u)$ $(j = 1, \ldots, p)$ and determine the quantity $u = a_1$, for which, at least for one j, the modulus $|\Phi_{0j}(u, u)| = C_0$ for $u = a_1$ and which increases with increasing u.

Second step. In the general case for $u = a_1$ the following system of equalities may take place:

$$\begin{aligned}\Phi_{0i_1}(\alpha_1,\alpha_1) &= \Phi_{0i_2}(\alpha_1,\alpha_1) = \ldots = \Phi_{0i_a}(\alpha_1,\alpha_1) = C_0\\ \Phi_{0i_{a+1}}(\alpha_1,\alpha_1) &= \Phi_{0i_{a+2}}(\alpha_1,\alpha_1) = \ldots = \Phi_{0i_b}(\alpha_1,\alpha_1) = -C_0\\ \Phi_{0i_{b+1}}(\alpha_1,\alpha_1) &= \Phi_{0i_{b+2}}(\alpha_1,\alpha_1) = \ldots = \Phi_{0i_c}(\alpha_1,\alpha_1) = C_0\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned} \tag{2.3}$$

The system of equalities (2.3) has N such rows. Each row contains at least one function. The functions in (2.3) are distributed so that their indices satisfy the inequality $i_1 < i_2 < \ldots < i_a < i_{a+1} < \ldots < i_b < i_{b+1} \ldots < i_c < \ldots$ For any function from an odd row, $\Phi_{0j}(u, u)$ increases, while for any function from an even row it decreases in some neighborhood to the right of the point $u = a_1$. If

$$|\Phi_{0,j-1}(\alpha_1,\alpha_1)| = |\Phi_{0j}(\alpha_1,\alpha_1)|, \qquad \Phi_{0,j-1}(u,u) \equiv \Phi_{0j}(u,u)$$

at some neighborhood to the right of the point $u = a_1$, then in the system of equalities (2.3) only $\Phi_{0,j-1}(a_1, a_1)$ is included.

Let us consider $\Phi_{0i_1}(z, z^*)$ for quantities z and z^* greater than a_1. From the relation

$$\Phi_{0i_1}(z, z^*) = C_0 \tag{2.4}$$

we will find at some neighborhood to the right of the point $z = a_1$ the function $z^* = \vartheta_{0i_1}(z)$. From the definition of $\Phi_{0i_1}(z, z^*)$ it follows that as long as $z + z^* \leqslant 2H$, the function $\vartheta_{0i_1}(z)$ is defined uniquely and at some neighborhood to the right of $z = a_1$, the inequality $z^* = \vartheta_{0i_1}(z) > z$ is valid.

If in the increase of z from a_1 for some a the inequality $\vartheta_{0i_1}(a) + a = 2H$ is valid, then for $z > a$ we let $\vartheta_{0i_1}(z) = \vartheta_{0i_1}(a)$.

Note that if in $\Phi_{0i_2}(z, z^*)$ the sum

$$\sum_{k=1}^{i_1}{}'' \delta_k(z^*) = 0 \text{ for } z > \alpha_1,\ z^* > \alpha_1, \text{ then } \vartheta_{0i_1}(z) = 2H - \alpha_1$$

Let us consider $\Phi_{0i_2}(z, z)$. From the relation

$$\Phi_{0i_2}(z, z^*) = C_0 \tag{2.5}$$

we find the function $z^* = \vartheta_{0i_2}(z)$.

If $\vartheta_{0i_2}(z) > \vartheta_{0i_1}(z)$ in the neighborhood to the right of $z = a_1$, then for subsequent constructions we will utilize only the function $\vartheta_{0i_2}(z)$.

If $\vartheta_{0i_2}(z) < \vartheta_{0i_1}(z)$, then from the relation

$$\Phi_{i_1 i_2}(z, z^*) \equiv \Phi_{0i_2}(z, z^*) - \Phi_{0i_1}(z, z^*) = \Phi_{i_1 i_2}(\alpha_1, \alpha_1) = 0 \qquad (2.6)$$

we find in some neighborhood to the right of $z = a_1$ the function $z^* = \vartheta_{i_1 i_2}(z)$ (this is possible since otherwise $\Phi_{0i_2}(a_1, a_{\ell})$ would be larger than C_0), and utilize further the functions $\vartheta_{0i_1}(z)$ and $\vartheta_{i_1 i_2}(z)$.

Assume now that in the first row of the system (2.3) all functionals up to Φ_{0i_ν} including $(i_\nu < i_a)$ have been considered and that for further construction the functions

$$\vartheta_{0i_\alpha}, \quad \vartheta_{i_\alpha i_\beta}, \quad \vartheta_{i_\beta i_\gamma}, \ldots, \vartheta_{i_\mu i_\nu} \qquad (2.7)$$

remain, where $i_\alpha < i_\beta < i_\gamma < \ldots < i_\mu < i_\nu$, and in the neighborhood to the right of $z = a_1$

$$\vartheta_{0i_\alpha}(z) > \vartheta_{i_\alpha i_\beta}(z) > \ldots > \vartheta_{i_\mu i_\nu}(z)$$

From the relation

$$\Phi_{0i_\nu}(z, z^*) = \Phi_{0i_\nu}(\alpha_1, \alpha_1) = C_0 \qquad (2.8)$$

we find the function $\vartheta_{0i_\nu}(z)$. If $\vartheta_{0i_\nu}(z)$ in the neighborhood to the right of $z = a$ is larger than $\vartheta_{i_\epsilon i_\rho}$ but is less than the preceding $\vartheta_{i_\epsilon i_\rho}$ function from (2.7), then for subsequent constructions we retain all functions from (2.7), preceding $\vartheta_{i_\epsilon i_\rho}(z)$, and the function $\vartheta_{i_\epsilon i_\nu}(z)$ which is found from the relation

$$\Phi_{i_\varepsilon i_\nu}(z, z^*) \equiv \Phi_{0i_\nu}(z, z^*) - \Phi_{0i_\varepsilon}(z, z^*) = 0 \qquad (2.9)$$

We will consider thus all Φ_{0i_k} from the first row of (2.3) and determine the corresponding functions $\vartheta_{ij}(z)$.

Let us consider now the second row of (2.3). From the relation

$$\Phi_{i_{a'} i_{a+1}}(z, z^*) \equiv \Phi_{0i_{a+1}}(z, z^*) - \Phi_{0i_{a'}}(z, z^*) = \Phi_{i_{a'} i_{a+1}}(\alpha_1, \alpha_1) = -2C_0 \quad (2.10)$$

$i_a \leqslant i_{a'} < i_{a+1}$ we find the function $z = \vartheta_{i_{a'} i_{a+1}}(z^*)$. The index $i_{a'}$

is determined as follows:

At some neighborhood to the right of the point a_1 the relation $\Phi_{0i_a}(u, u) = \Phi_{0j}(u, u)$ is satisfied identically with respect to the argument u for all $\Phi_{0j}(u, u)$ and $i_a \leqslant j \leqslant i_{a'}$, while for the index $j > i_{a'}$, but smaller than i_{a+1}, the inequality $\Phi_{0i_a}(u, u) > \Phi_{0j}(u, u)$ is satisfied in any arbitrarily small neighborhood to the right (Fig. 3) of the point a_1.

Next consider $\Phi_{0i_{a+2}}(z, z^*)$. From the relation

$$\Phi_{i_{a'}i_{a+2}}(z, z^*) \equiv \Phi_{0i_{a+2}}(z, z^*) - \Phi_{0i_{a'}}(z, z^*) = -2C_0 \tag{2.11}$$

we find $z = \vartheta_{i_{a'}i_{a+2}}(z^*)$. If at some neighborhood to the right of $z^* = a_1$ the function $\vartheta_{i_{a'}i_{a+2}}(z^*)$ is greater than $\vartheta_{i_{a'}i_{a+1}}(z^*)$ we retain one function $\vartheta_{i_{a'}i_{a+2}}(z^*)$. In the opposite case we find from the relation

$$\Phi_{i_{a+1}i_{a+2}}(z, z^*) \equiv \Phi_{0i_{a+2}}(z, z^*) - \Phi_{0i_{a+1}}(z, z^*) = 0 \tag{2.12}$$

the function $z = \vartheta_{i_{a+1}i_{a+2}}(z^*)$. Further, we will consider functions $\vartheta_{i_{a'}i_{a+1}}(z^*)$ and $\vartheta_{i_{a+1}i_{a+2}}(z^*)$, etc.

Using the analogous reasoning, it is possible from row to row to consider all functions included in the system of equalities (2.3) and to construct at some neighborhood to the right of the point $u = a_1$ the system of functions $z = \vartheta_{ij}(u)$ and $z^* = \vartheta_{ij}(u)$.

Let us locate all $\Phi_{ij}(z, z^*)$, used for the determination of the system of functions ϑ_{ij}, in the order of ascending indices

$$\Phi_{i_1j_1}(z, z^*),\quad \Phi_{i_2j_2}(z, z^*), \ldots, \Phi_{i_mj_m}(z, z^*) \tag{2.13}$$

Substituting in each element of (2.13) the functions $\vartheta_{ij}(u)$ in place of the corresponding arguments, we will obtain the functions $\Phi_{ij}(\vartheta_{ij}(u), u)$ or $\Phi_{ij}(u, \vartheta_{ij}(u))$. If in (2.13) $j_k < j_{k+1}$, then we will add to the sequence (2.13) the function $\Phi_{j_kj_{k+1}}(u, u)$, inserting it between $\Phi_{i_kj_k}$ and $\Phi_{i_{k+1}j_{k+1}}$. The result is the sequence

$$\Phi_{j_0j_1},\quad \Phi_{j_1j_2},\quad \Phi_{j_2j_3}, \ldots, \Phi_{j_{p-1}j_p} \tag{2.14}$$

As was pointed out above, to each $\Phi_{j_{k-1}j_k}$ from (2.14) corresponds a function $\vartheta_{j_{k-1}j_k}(u)$. These functions form the sequence

$$\vartheta_{j_0j_1},\ \ \vartheta_{j_1j_2},\ \ \vartheta_{j_2j_3},\ldots,\vartheta_{j_{p-1}j_p} \tag{2.15}$$

Let $j_k \leqslant j \leqslant j_{k+1}$, where j_k and j_{k+1} are indices of some function $\Phi_{j_kj_{k+1}}(u)$ from (2.14).

If $\Phi_{j_kj_{k+1}}(u) \equiv \Phi_{j_kj_{k+1}}(u,\ \vartheta_{j_kj_{k+1}}(u))$, then let $\Phi_j(u)$ denote the function

$$\Phi_j(u)=\sum_{i=1}^{k}\Phi_{j_{i-1}j_i}(u)+\Phi_{j_kj}(u,\ \vartheta_{j_kj_{k+1}}(u)) \tag{2.16}$$

If

$$\Phi_{j_kj_{k+1}}(u)\equiv\Phi_{j_kj_{k+1}}(\vartheta_{j_kj_{k+1}}(u),\ u)$$

then

$$\Phi_j(u)=\sum_{i=1}^{k}\Phi_{j_{i-1}j_i}(u)+\Phi_{j_kj}(\vartheta_{j_kj_{k+1}}(u),\ u) \tag{2.17}$$

If $j = j_k$ then the functions $\Phi_{j_k} = \sum_{i=1}^{k}\Phi_{j_{i-1}j_i}(u)$ form a sequence

$$\Phi_{j_1}(u),\ \ \Phi_{j_2}(u),\ldots,\Phi_{j_p}(u) \tag{2.18}$$

Their indices form a sequence

$$j_1,\ \ j_2,\ldots,j_p \tag{2.19}$$

All elements in (2.19) can be divided into N groups according to the number of rows in (2.3). In each group the elements j_k are distributed in the order of ascending index k, while the indices in the elements of the ith group are smaller than the indices of the $i+1$ group. Let j_k belong to the ith group from (2.19), then Φ_{j_k}, $\Phi_{j_{k-1}j_k}$, $\vartheta_{j_{k-1}j_k}$ belong to the ith groups of the sequences (2.18), (2.14), (2.15). If $\vartheta_{j_kj_{k+1}}$ belongs to the ith group and is determined from the relation

$$\Phi_{j_kj_{k+1}}(z,\ z^*)=\Phi_{j_kj_{k+1}}(\alpha_1,\ \alpha_1)$$

as functions of z, i.e. $\vartheta_{j_kj_{k+1}} = \vartheta_{j_kj_{k+1}}(z)$, then the remaining functions of this group are also determined from the corresponding relations as functions of z. If, however, $\vartheta_{j_k j_{k+1}}$ is defined as the function of z^*, the remaining functions of this group are likewise defined. From the scheme of construction of functions $\vartheta_{j_k j_{k+1}}$ it follows that:

1) At some neighborhood to the right of the point $u = a_1$ for all u the functions $\vartheta_{j_k j_{k+1}}(u)$ from one group vary monotonically with the increase of index k, while at the same time, the difference $\vartheta_{j_k j_{k+1}}(u) - \vartheta_{j_{k-1} j_k}(u)$ does not change sign with the increase in u in the region of this neighborhood.

2) In this neighborhood $\vartheta_{j_k j_{k+1}}(u) \geqslant u$ for any k.

3) All $\Phi_{j_k}(u)$ belonging to this group are identically equal to each other at the neighborhood to the right of $u = a_1$.

Note that

$$\begin{aligned} \Phi_{j_k j_{k+1}}(u, \vartheta_{j_k j_{k+1}}(u)) &= \sum_{i=j_k+1}^{j_{k+1}}{}' \delta_i(u) - \sum_{i=j_k+1}^{j_{k+1}}{}'' \delta_i(\vartheta_{j_k j_{k+1}}(u)) \\ \Phi_{j_k j_{k+1}}(\vartheta_{j_k j_{k+1}}(u), u) &= \sum_{j=j_k+1}^{j_{k+1}}{}' \delta_i(\vartheta_{j_k j_{k+1}}(u)) - \sum_{i=j_k+1}^{j_{k+1}}{}'' \delta_i(u) \end{aligned} \tag{2.20}$$

where each sum is a monotonically increasing function of the argument u.

Let us denote by E_u^+ the system of intervals defined by the sum of the first components of all functions from (2.14), and by E_u^- the system of intervals defined by the sum of the second components of all functions from (2.14). From (2.20) it follows that if $a_1 < u_1 < u_2$ then

$$E_{a_1}^+ \subset E_{u_1}^+ \subset E_{u_2}^+, \qquad E_{a_1}^- \subset E_{u_1}^- \subset E_{u_2}^- \tag{2.21}$$

Let us define now the function $\phi_u(t)$.

$$\begin{gathered} \varphi_u(t) = M_1, \quad \text{if } t \in E_u^+; \qquad \varphi_u(t) = -M_1, \quad \text{if } t \in E_u^- \\ \varphi_u(t) = 0, \qquad \text{if } t \in [0, T] - E_u^+ - E_u^- \end{gathered}$$

From (3) above it follows that $\phi_u(t)$ belongs to class A of the functions considered; it is easy to show that $Y(\phi_u)$ increases with the increase in u. All this is valid at some neighborhood to the right of the point $u = a_1$. The second step ends when $u = a_2$, if for this value of u at least one of the following four cases occurs:

A) It may be that for $y = a_2$

$$\alpha_2 + \vartheta_{j_k j_{k+1}}(\alpha_2) = 2H$$

for all functions from (2.15). In this case we consider that $\phi_{a_2}(t)$ coincides with the maximal function $\phi_m(t)$, the construction of which is thus completed.

B) Let $j_k < j < j_{k+1}$ where j_k and j_{k+1} are the elements from (2.19), then it may be that $|\Phi_j(a_2)| = C_0$ and $|\Phi_j(u)|$ increases to the right of $u = a_2$. These relationships can take place simultaneously for several j lying between j_k and j_{k+1} and for several values of the index k.

C) For one or for several values of k and for $u = a_2$ the difference $\vartheta_{j_k j_{k+1}}(u) - \vartheta_{j_{k-1} j}(u)$ changes sign. It is assumed that $\vartheta_{j_{k-1} j_k}(u)$ and $\vartheta_{j_k j_{k+1}}(u)$ belong to one group from (2.15).

D) For $u = a_2$ the difference $\vartheta_{j_k j_{k+1}}(u) - u$ changes sign for one or for several values of k. If for $u = a_2$ at least one of the cases (B), (C) or (D) takes place, then in order to define $\phi_u(t)$ for $u > a_2$ it is necessary to transform and augment the sequences (2.14), (2.15), (2.18), (2.19) according to the following rules.

Let us assume case (B). Depending on the sign of $\Phi_j(a_2)$, $\Phi_{j_k}(a_2)$, $\Phi_{j_{k+1}}(a_2)$, there can be eight cases which can be represented by the following scheme:

	1	2	3	4	5	6	7	8
$\Phi_{j_k} =$	C_0	C_0	C_0	C_0	$-C_0$	$-C_0$	$-C_0$	$-C_0$
$\Phi_j =$	C_0	$-C_0$	$+C_0$	$-C_0$	$+C_0$	$-C_0$	$+C_0$	$-C_0$
$\Phi_{j_{k+1}} =$	C_0	C_0	$-C_0$	$-C_0$	C_0	C_0	$-C_0$	$-C_0$

Consider $\Phi_{j_k j_{k+1}}(z + z^*)$. For $u = a_2$ there will be

$$\text{either (a)} \quad z = \alpha_2, \quad z^* = \vartheta_{j_k j_{k+1}}(\alpha_2)$$
$$\text{or (b)} \quad z^* = \alpha_2, \quad z = \vartheta_{j_k j_{k+1}}(\alpha_2)$$

In case of (a) at some neighborhood to the right of $z = a_2$ we find $z^* = \vartheta_{j_k j}(z)$ from the relation

$$\Phi_{j_k j}(z, z^*) = \Phi_j(\alpha_2) - \Phi_{j_k}(\alpha_2)$$

From the relation $\Phi_{j j_{k+1}}(z, z^*) = \Phi_{j_{k+1}}(a_2) = \Phi_j(a_2)$ we find

$z^* = \vartheta_{j_k j}(z)$; and in the sequence (2.14) in place of $\Phi_{j_k j_{k+1}}(u, \vartheta_{j_k j_{k+1}}(u))$ we substitute $\Phi_{j_k j}(u, \vartheta_{j_k j}(u))$ and $\Phi_{j j_{k+1}}(u, \vartheta_{j j_{k+1}}(u))$; in (2.15) in place of $\vartheta_{j_k j_{k+1}}$ we substitute $\vartheta_{j_k j}(u)$ and $\vartheta_{j j_{k+1}}(u)$; in (2.18) and (2.19) between Φ_{j_k} and $\Phi_{j_{k+1}}$, j_k and j_{k+1} we insert correspondingly Φ_j and j.

In case of (b) we find from the same relationships the function $z = \vartheta_{j_k j}(z^*)$ and $z = \vartheta_{j j_{k+1}}(z^*)$ and repeat the same procedure as for (a).

If between Φ_{j_k} and $\Phi_{j_{k+1}}$ for $u = a_2$ there are valid several relations of the type $|\Phi_j(a_2)| = C_0$, we then utilize the same method which was used at the start of the second step for determination of $\vartheta_{ij}(u)$, and introduce into the sequences (2.14), (2.15), (2.18), (2.19) corresponding corrections.

Assume that case (C) occurs and let, for example, $\vartheta_{j_{k-1} j_k}$ be determined from the relation $\Phi_{j_{k-1} j_k}(z, z^*) = \Phi_{j_{k-1} j_k}(a_1, a_1)$ as a function of z. Then from the relation $\Phi_{j_{k-1} j_{k+1}}(z, z^*) = \Phi_{j_{k+1}}(a_1) = \Phi_{j_{k-1}}(a_1)$ we find the function $z^* = \vartheta_{j_{k-1} j_{k+1}}(z)$ in the neighborhood to the right of $z = a_2$.

In (2.14) in place of $\Phi_{j_{k-1} j_k}(u, \vartheta_{j_{k-1} j_k}(u))$ and $\Phi_{j_k j_{k+1}}(u, \vartheta_{j_k j_{k+1}}(u))$ we substitute $\Phi_{j_{k-1} j_{k+1}}(u, \vartheta_{j_{k-1} j_{k+1}}(u))$; in (2.15) in place of $\vartheta_{j_{k-1} j_k}$ and $\vartheta_{j_k j_{k+1}}$ we substitute $\vartheta_{j_{k-1} j_{k+1}}$; in (2.18) and (2.19) we strike out Φ_{j_k} and j_k.

Assume that case (D) occurs, i.e. $\vartheta_{j_{k-1} j_k}(u) - u$ changes sign for $u = a_2$. Case (D) can have several versions:

1) In (2.15) $\vartheta_{j_{k-1} j_k}$ is the function furthest to the right, or to the right of it is only $\vartheta_{j_k j_{k+1}} \equiv u$.

In this case in (2.15) in place of $\vartheta_{j_{k-1} j_k}$ and $\vartheta_{j_k j_{k+1}}$ we substitute $\vartheta_{j_{k-1} j_{k+1}}(u) \equiv u$, in (2.14) in place of $\Phi_{j_{k-1} j_k}$ and $\Phi_{j_k j_{k+1}}$ we substitute $\Phi_{j_{k-1} j_{k+1}}(u, u)$; in (2.18) and (2.19) we strike out Φ_{j_k} and j_k.

2) The function $\vartheta_{j_{k-1} j_k}$ belongs to the ith group from (2.15), and to the right of it in this group can be only the function $\vartheta_{j_k j_{k+1}}(u) \equiv u$;

let, for example

$$\Phi_{j_{k-1} j_k}(u,\ \vartheta_{j_{k-1} j_k}(u)) = \Phi_{j_{k-1} j_k}(\alpha_1,\ \alpha_1)$$

Then from the relation

$$\Phi_{j_{k-1} j_{k+1}}(z,\ z^*) = \Phi_{j_{k-1} j_{k+1}}(\alpha_1,\ \alpha_1)$$

we find $z = \vartheta_{j_{k-1} j_{k+1}}(z^*)$; substitute in (2.15) in place of $\vartheta_{j_{k-1} j_k}(u)$ and $\vartheta_{j_k j_{k+1}}(u)$ the function $\vartheta_{j_{k-1} j_{k+1}}(u)$; in (2.14) substitute $\Phi_{j_{k-1} j_{k+1}}$ in place of $\Phi_{j_{k-1} j_k}$ and $\Phi_{j_k j_{k+1}}$, etc.

3) The function $\vartheta_{j_{k-1} j_k}$ is the furthest to the left in the ith group from (2.15). Let, for example

$$\Phi_{j_{k-1} j_k}(\vartheta_{j_{k-1} j_k}(u),\ u) = \Phi_{j_{k-1} j_k}(\alpha_1,\ \alpha_1)$$

If $\vartheta_{j_{k-2} j_{k-1}}(u) \equiv u$, then from the relation $\Phi_{j_{k-2} j_k}(z,\ z^*) = \Phi_{j_{k-2} j_k}(a_1, a_1)$ we find the function $z = \vartheta_{j_{k-2} j_k}(z^*)$; in (2.15) we substitute $\vartheta_{j_{k-2} j_k}(u)$ in place of $\vartheta_{j_{k-2} j_{k-1}}$ and $\vartheta_{j_{k-1} j_k}$; in (2.14) we substitute $\Phi_{j_{k-2} j_k}(u,\ \vartheta_{j_{k-2} j_k}(u))$ in place of $\Phi_{j_{k-2} j_{k-1}}(u,\ u)$ and $\Phi_{j_{k-1} j_k}(\vartheta_{j_{k-1} j_k}(u),\ u)$, etc.

If $\vartheta_{j_{k-2} j_{k-1}}(u) \not\equiv u$ then, from the relation $\Phi_{j_{k-1} j_k}(z,\ z^*) = \Phi_{j_{k+1} j_k}(a_1,\ a_1)$ we will find $z = \vartheta_{j_{k-1} j_k}(z^*)$ and in (2.15) we substitute $\vartheta_{j_{k-1} j_k}(u)$ in place of the previous function with the same index; in (2.14) substitute $\Phi_{j_{k-1} j_k}(u,\ \vartheta_{j_{k-1} j_k}(u))$ in place of $\Phi_{j_{k-1} j_k}(\vartheta_{j_{k-1} j_k}(u),\ u)$ (the functions $\vartheta_{j_{k-1} j_k}(u)$ are different in the last two expressions). Thus all cases have been considered which can occur when $u = a_2$.

The thus-transformed sequences (2.14), (2.15), (2.18), (2.19) form new sequences which can differ from the original ones only by the number of groups. Utilizing these new sequences it is possible to determine $\phi_u(t)$ at some neighborhood to the right of $u = a_2$. If for $u = a_3$ case (A) applies, then $\phi_{a_3}(t)$ coincides with the maximal function; if however, for $u = a_3$ any one of the cases (B), (C) or (D) applies, it is necessary to transform the new sequences obtained as shown above and to determine $\phi_u(t)$ for $u > a_3$. The construction of $\phi_m(t)$ will be completed if for any one of the analogous steps for $0 < u \leqslant H$ the case (A) will apply. Since

$F'(t)$ is bounded on $[0, T]$ there will be a finite number of steps necessary in order to construct the maximal function. Thus, the algorithm for constructing $\phi_m(t)$ has been indicated.

3. We will prove now that the previously constructed maximal function $\phi_m(t)$ indeed gives the largest value to the functional (1.7). Let $\phi_r(t)$ be an arbitrary function of class A; we will prove that

$$Y(\varphi_m) - Y(\varphi_r) \geqslant 0 \tag{3.1}$$

From Section 2 it follows that if on $[0, T]$ there is no such point t_0 for which

$$|f_m(t_0)| = \left|\int_0^{t_0} \varphi_m(t)\, dt\right| = M_0, \quad \text{then} \quad \varphi_m(t) = M_1 \operatorname{sign} F(t)$$

It follows immediately from (1.7) that in this case $\phi_m(t)$ yields the largest value of the functional $Y(\phi)$.

Let us consider the most general case. From the construction algorithm for $\phi_m(t)$ it follows that for $f_m(t)$ there can be the following relations

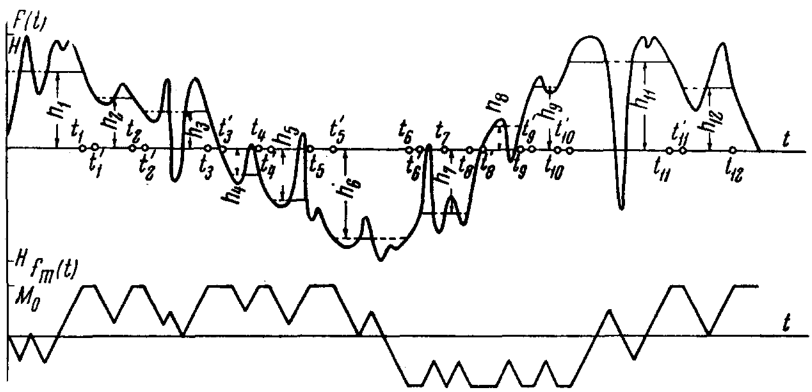

Fig. 4.

(see Fig. 4, where on the intervals denoted by the solid line $\phi_m = M_1$, dotted line $\phi_m = -M_1$; outside of these intervals $\phi_m = 0$; $t_3 = t_{i_1}$, $t_5 = t_{i_2}$, $t_8 = t_{i_3}$, $t_{10} = t_{i_4}$, $t_{12} = t_{i_5}$):

1) At points $t_1, t_2, \ldots, t_{i_1}$ the equalities

$$f_m(t_1) = f_m(t_2) = \ldots = f_m(t_{i_1}) = M_0 \tag{3.2}$$

are fulfilled, while at some neighborhood to the left of each of these points $f_m(t) < M_0$. There are i_1 numbers $h_1 \geqslant h_2 \geqslant \dots \geqslant h_{i_1} \geqslant 0$ such that

$$\varphi_m(t) = M_1 \operatorname{sign}(F(t) - h_1) \tag{3.3}$$

The function $\phi_m(t) = M_1$ sign $(F(t) - h_2)$ on the interval (t_1, t_2), with the exception of the interval $(t_1, t_1') \in (t_1, t_2)$ where t_1' is nearest to t_1 point of intersection of function $x = F(t)$ with the straight line $x = h_2$; on this interval $\phi_m(t) = 0$.

Function $\phi_m(t) = M_1$ sign $(F(t) - h_3)$ on the interval (t_2, t_3), with the exception of the interval (t_2, t_2') where t_2' is nearest to t_2 point of intersection of function $x = F(t)$ with the straight line $x = h_3$; on this interval $\phi_m(t) = 0$, etc.

2) At points $t_{i_1+1}, t_{i_1+2} \dots t_{i_2}$ the following equalities are fulfilled:

$$f_m(t_{i_1+1}) = f_m(t'_{i_1+2}) = \dots f_m(t_{i_2}) = M_0 \qquad (t_{i_1} < t_{i_1+1} < t_{i_1+2} < \dots < t_{i_2}) \tag{3.4}$$

With regard to this group of equalities and determination of the function $\phi_m(t)$ on the corresponding intervals, it is necessary to repeat literally everything as in the previous case, with the only difference that

$$0 > h_{i_1+1} \geqslant h_{i_1+2} \geqslant \dots \geqslant h_{i_2}$$

3) At points $t_{i_2+1}, t_{i_2+2}, \dots, t_{i_3}$ the following equalities are fulfilled:

$$f_m(t_{i_2+1}) = f_m(t_{i_2+2}) = \dots = f_m(t_{i_3}) = -M_0 \tag{3.5}$$

where at the same time $f_m(t) > -M_0$ at some neighborhood to the left of points $t_{i_2+1}, t_{i_2+2}, \dots, t_{i_3}$.

Function $\phi_m(t)$ is defined on the intervals $(t_{i_2+1}, t_{i_2+2}) \dots (t_{i_3-1}, t_{i_3})$ in the same way as previously, but here

$$0 > h_{i_3} \geqslant h_{i_3-1} \geqslant \dots \geqslant h_{i_2+1} \tag{3.6}$$

4) At points $t_{i_3+1}, t_{i_3+2}, \dots, t_{i_4}$ the following equalities are fulfilled:

$$f_m(t_{i_3+1}) = f_m(t_{i_3+2}) = \dots = f_m(t_{i_4}) = -M_0 \tag{3.7}$$

Everything is determined as in (3), the only exception being that

$$0 \leqslant h_{i_3+1} \leqslant h_{i_3+2} \leqslant \ldots h_{i_4} \tag{3.8}$$

5) At points $t_{i_4+1}, t_{i_4+2}, \ldots, t_{i_5}$ the following equalities are fulfilled:

$$f_m(t_{i_4+1}) = f_m(t_{i_4+2}) = \ldots f_m(t_{i_5}) = M_0 \tag{3.9}$$

Here one should repeat literally everything that was said in (1). Next follow equalities (6) coinciding with the equalities in (2), etc.

Figure 4 shows such a function $f_m(t)$. Let us show that

$$\int_0^{t_{i_1}'} F(t)(\varphi_m - \varphi_r)\,dt \geqslant 0 \tag{3.10}$$

where t_{i_1}' is the root of equation $F(t) = 0$ nearest to t_{i_1} from the right.

By Δ_j, Δ_j', Δ_j'' ($j = 1, \ldots, i_1$) let us denote the following systems of intervals belonging to $(0, t_{i_1}')$ (see Fig. 5, where the intervals from (t_1', t_2') are indicated with double lines forming Δ_2; (t_2, t_2') is Δ_2'', all other intervals from (t_1', t_2') form Δ_2'):

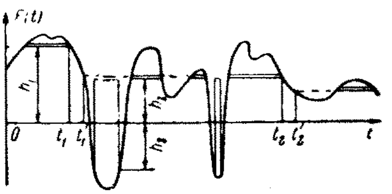

Fig. 5.

a) $\Delta_j + \Delta_j' + \Delta_j'' = (t_{j-1}', t_j)$.

b) If $t \in \Delta$ then $|F(t)| > h_j$ and $\phi_m(t) = M_1$ sign $F(t)$; if $t \in \Delta j'$ then $|F(t)| < h_j$ and $\phi_m(t) = -M_1$; if $t \in \Delta_j''$ then $h_j \geqslant F(t) \geqslant h_{j+1}$ and $\phi_m(t) = 0$ (Δ_j'' is the interval (t_j, t_j')).

From above it follows that

$$\int_0^{t_{i_1}} F(t)(\varphi_m - \varphi_r)\,dt$$
$$= \sum_{j=1}^{i_1} \left\{ \int_{\Delta_j} F(\varphi_m - \varphi_r)\,dt + \int_{\Delta_j'} F(\varphi_m - \varphi_r)\,dt + \int_{\Delta_j''} F(-\varphi_r)\,dt \right\}$$

The integrals on the right-hand side are evaluated along the intervals contained in Δ_j, Δ_j', Δ_j''. Utilizing the generalized theorem about the mean, it is possible to show that

$$\int_{\Delta_j} F(\varphi_m - \varphi_r)\,dt = |F(a_j)| \int_{\Delta_j} |\varphi_m - \varphi_r|\,dt \qquad (a_j \in \Delta_j)$$
$$\int_{\Delta_j'} F(\varphi_m - \varphi_r)\,dt = F(b_j) \int_{\Delta_j'} (\varphi_m - \varphi_r)\,dt \qquad (b_j \in \Delta_j')$$
$$\int_{\Delta_j''} F(-\varphi_r)\,dt = F(c_j) \int_{\Delta_j''} (-\varphi_r)\,dt \qquad (c_j \in \Delta_j'')$$

Therefore

$$\int_0^{t_{i_1}'} F(\varphi_m - \varphi_r)\,dt \tag{3.11}$$
$$= \sum_{j=1}^{i_1} \left\{ |F(a_j)| \int_{\Delta_j} |\varphi_m - \varphi_r|\,dt + F(b_j) \int_{\Delta_j'} (\varphi_m - \varphi_r)\,dt + F(c_j) \int_{\Delta_j''} (-\varphi_r)\,dt \right\}$$

Note that $\int_{\Delta_j'} (\phi_m - \phi_r)\,dt \leqslant 0$, since $\phi_m(t) = -M_1$ on Δ_j'. It is obvious that

$$\sum_{j=1}^{k} \left\{ \int_{\Delta_j} (\varphi_m - \varphi_r)\,dt + \int_{\Delta_j'} (\varphi_m - \varphi_r)\,dt + \int_{\Delta_j''} (-\varphi_r)\,dt \right\} \geqslant 0 \qquad (k = 1, \ldots i_1)$$

Let for some $k < i_1$ be shown that

$$\sum_{j=1}^{k} \left\{ |F(a_j)| \int_{\Delta_j} |\varphi_m - \varphi_r|\,dt + F(b_j) \int_{\Delta_j'} (\varphi_m - \varphi_r)\,dt + F(c_j) \int_{\Delta_j''} (-\varphi_r)\,dt \right\} \geqslant 0$$

We will show that a similar inequality takes place for $k + 1$. Indeed, let

$$\int_{\Delta_{k+1}''} (-\varphi_r(t))\,dt < 0$$

Then if

$$\int_{\Delta_{k+1}} (\varphi_m - \varphi_r)\,dt > \left| \int_{\Delta'_{k+1}} (\varphi_m - \varphi_r)\,dt + \int_{\Delta''_{k+1}} (-\varphi_r)\,dt \right|$$

it follows from the inequality $|F(a_{k+1})| > \max\,(|F(b_{k+1})|,\ F(C_{k+1}))$ that

$$|F(a_{k+1})| \int_{\Delta_{k+1}} |\varphi_m - \varphi_r|\,dt + F(b_{k+1}) \int_{\Delta'_{k+1}} (\varphi_m - \varphi_r)\,dt + \\ + F(c_{k+1}) \int_{\Delta''_{k+1}} (-\varphi_r)\,dt \geqslant 0$$

If

$$\int_{\Delta_{k+1}} (\varphi_m - \varphi_r)\,dt < \left| \int_{\Delta'_{k+1}} (\varphi_m - \varphi_r)\,dt + \int_{\Delta''_{k+1}} (-\varphi_r)\,dt \right|$$

then from the inequalities

$$\sum_{j=1}^{k+1} \left\{ \int_{\Delta_j} (\varphi_m - \varphi_r)\,dt + \int_{\Delta_j'} (\varphi_m - \varphi_r)\,dt + \int_{\Delta_j''} (-\varphi_r)\,dt \right\} \geqslant 0$$

$$\max\,(|F(b_{k+1})|,\, F(c_{k+1})) < \min_{j \leqslant k}\,(|F(a_j)|,\ F(c_j))$$

it follows that

$$\sum_{j=1}^{k+1} \left\{ |F(a_j)| \int_{\Delta_j} |\varphi_m - \varphi_r|\,dt + F(b_j) \int_{\Delta_j'} (\varphi_m - \varphi_r)\,dt + F(c_j) \int_{\Delta_j''} (-\varphi_r)\,dt \right\} \geqslant 0$$

The inequality

$$|F(a_1)| \int_{\Delta_1} |\varphi_m - \varphi_r|\,dt + F(b_1) \int_{\Delta_1'} (\varphi_m - \varphi_r)\,dt + F(c_1) \int_{\Delta_1''} (-\varphi_r)\,dt \geqslant 0$$

follows from

$$\int_{\Delta_1 + \Delta_1' + \Delta_1''} (\varphi_m - \varphi_r)\,dt \geqslant 0, \qquad |F(a_j)| > \max\,(|F(b_1)|,\ F(c_1))$$

Thus, the inequality (3.10) is proved. It is proved analogously as well as when

$$\int_{\Delta''_{k+1}} (-\varphi_r)\,dt \geqslant 0$$

Then one considers the interval (t_{i_1}', t_{i_3}'), where t_{i_3}' is the root of equation $F(t) = 0$ nearest to t_{i_3} from the right. It can be shown that

$$\int_{t_{i_1}'}^{t_{i_3}'} F(\varphi_m - \varphi_r) \geqslant 0 \qquad (3.12)$$

The proof of this inequality is analogous to that of (3.10) and is therefore omitted. Note only that for the proof one utilizes the inequalities of the form (3.8) and the inequality

$$\int_{t_{i_1}'}^{t_j} (\varphi_m - \varphi_r)\, dt \leqslant 0 \qquad (j = i_2 + 1,\ i_2 + 2, \ldots, i_3)$$

Next, one considers the interval (t_{i_3}', t_{i_5}') and proves the inequality

$$\int_{t_{i_3}'}^{t'_{i_5}} F(\varphi_m - \varphi_r)\, dt \geqslant 0 \qquad \text{etc.} \qquad (3.13)$$

All these inequalities are proved analogously to (3.10). Since the number of such intervals is finite this proves the validity of (3.1).

It may be that the limiting equalities for $f_m(t)$ will start not from case (1), as was shown above, but from any of the following cases. Furthermore, case (1) may be followed immediately by case (3), etc. However, the proof of (3.1) remains unchanged.

Example. In the equation $\ddot{y} + 2\dot{y} + 100y = f(t)$ for $T = 1$ $y_{max}(1) = 4k$ under the condition $|f(t)| \leqslant 100k$; $y_{max}(1) = 3.3k$ under the condition $|f(t)| \leqslant 100k$; $|f'(t)| \leqslant 830k$.

Note 1. The algorithm for constructing $\phi_m(t)$ presented above can also be applied in the case of a linear difference equation; in contrast to the presented case, however, $\phi_m(t)$ may not be the only function giving a maximum to the solution at time T (see [5]).

Note 2. If only a few steps are required for the determination of $\phi_m(t)$, this can easily be accomplished graphically. In the general case it is not difficult to program the evaluation of $\phi_m(t)$ and $Y_m(t)$ on an electronic digital computer.

BIBLIOGRAPHY

1. Bulgakov, B.V., O nakoplenii vozmushchenii v lineinykh kolebatel'nykh sistemakh s postoiannymi paramterami (On the accumulation of disturbances in linear oscillating systems with constant parameters). *Dokl. Akad. Nauk SSSR* Vol. 51, No. 5, 1946.

2. Bulgakov, B.V. and Kuzovkov, N.T., O nakoplenii vozmushchenii v lineinykh sistemakh s peremennymi parametrami (On the accumulation of disturbances in linear systems with variable parameters). *PMM* Vol. 14, No. 1, 1950.

3. Roitenberg, Ia.N., O nakoplenii vozmushchenii v nestatsionarnykh lineinykh impul'snykh sistemakh (On the accumulation of disturbances in nonstationary linear impulsive systems). *PMM* Vol. 22, No. 4, 1958.

4. Gnoenskii, L.S., O nakoplenii vozmushchenii v lineinykh sistemakh (On the accumulation of disturbances in linear systems). *Nauchn. Dokl. Vyssh. Shkoly Fiz.-matem. Nauki* No. 1, 1959.

5. Gnoenskii, L.S., O nakoplenii vozmushchenii v nestatsionarnykh lineinykh impul'snykh sistemakh (On the accumulation of disturbances in nonstationary linear impulsive systems). *PMM* Vol. 23, No. 6, 1959.

6. Birch, B.J. and Jackson, R., The behavior of linear systems with inputs satisfying certain bounding conditions. *Journal of Electronics and Control* Series I, Vol. 6, No. 4, Apr. 1959.

Translated by V.C.